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Nuclear Physics **35** (1962) 686—693; © *North-Holland Publishing Co., Amsterdam*

FIELD PROPAGATORS SINGULAR AT ZERO COUPLING

J. A. SWIECA †

Department of Physics, University of São Paulo, Brazil ††

Received 10 October 1961

Abstract: The asymptotic equivalence, for vanishing coupling, between axiomatic and ghost propagators is investigated. The physical implications of such an equivalence for the singular structure of a field theory and for the high energy properties of field propagators are discussed.

1. Introduction

This paper is to be considered as a direct continuation of a preceding one[1]) ††† in which the implications of a singular behaviour of quantum field theories at zero coupling were discussed.

In this note we shall show under which conditions a non-axiomatic propagator with a ghost pole ('ghost propagator') coincides asymptotically for vanishing coupling with any one of a class of axiomatically correct propagators.

The motivation for this analysis lies in the fact, recently pointed out by several authors [1-3]), that the subtraction of the ghost pole from the totally iterated bubble approximation to meson and photon propagators leads to an axiomatically correct approximate propagator having the same power series expansion as the ghost propagator.

We discuss simultaneously both the case in which the ghost propagator arises from an approximation made to an axiomatically correct one and the case in which already the exact one turns out to present a ghost pole.

The implications of the asymptotic equivalence for the singular structure of the theory and the high energy behaviour of the propagators will also be discussed.

The physical ideas underlying the present work have been discussed in (I).

The following notation will be used: $k^2 = \mathbf{k}^2 - k_4^2$ is to be considered a complex variable and the physical propagators, self-energies etc. are defined in the usual way as boundary values of the corresponding functions of k^2 (i.e. for a real k^2 the substitution $k^2 \to k^2 - i\varepsilon$ should be made).

† Present address: Max-Planck-Institut für Physik und Astrophysik, München, Germany.

†† Work done under the auspices of the Brazilian National Research Council and the International Atomic Energy Agency.

††† Ref. [1]) is hereafter referred to as (I).

2. Asymptotic Requirement

In this section, we determine the class of axiomatic boson propagators which have a given ghost propagator as an asymptotic representation for vanishing coupling ($g^2 \to +0$).

Let the renormalized ghost propagator of the boson be

$$\Delta'_F = 1/[(k^2+m^2)(1-F(g^2, k^2))], \tag{1}$$

where $F(g^2, k^2)$ is analytic in the complex k^2 plane cut from $-\infty$ to $-a^2$. Here, $a^2 \geqq m^2$, m being the renormalized boson mass, and $F(g^2, -k^2+i\varepsilon) = F^*(g^2, -k^2-i\varepsilon)$ for $g^2 > 0$. The function F has no singularities along the cut and its Cauchy integral around the branch point a^2 tends to zero with vanishing radius. Since the propagator should tend to a free one as the coupling constant tends to zero and m is the renormalized mass, we require that $F(0, k^2) = 0$ and $F(g^2, -m^2) = 0$.

Let us assume that $F(g^2, k^2)$ is an analytic function of g^2 in a certain domain (independent of k^2) around zero.

The presence of a ghost pole k_0^2 in (1) implies $1-F(g^2, k_0^2) = 0$ with k_0^2 real for $g^2 > 0$ and $k_0^2 > -m^2$. We consider the case of only one ghost state and consequently restrict ourselves to simple zeros of the above equation.

Suppose that $\Delta'_F \to 0$ faster than $1/k^2$ as $k^2 \to \infty$; this hypothesis might be weakened to $\Delta'_F \to 0$ as $k^2 \to \infty$, without altering the majority of our results. The above hypotheses are suggested by the results of perturbation theory; they are satisfied by a finite-order renormalized approximation to F in eq. (1).

The requirement of asymptotic equivalence between the axiomatic propagator $\bar{\Delta}'_F$ and the ghost propagator Δ'_F implies of course that their difference, denoted by $R(g^2, k^2)$, has zero asymptotic expansion. The axiomatic propagator

$$\bar{\Delta}'_F = \Delta'_F + R(g^2, k^2) \tag{2}$$

satisfies the Lehmann [4]) representation †

$$\bar{\Delta}'_F = 1/(k^2+m^2) + \int_{a^2}^{\infty} du^2 K(g^2, u^2)/(k^2+u^2), \tag{3}$$

with $K(g^2, u^2) \geqq 0$ for $g^2 > 0$.

From eqs. (1)-(3) it follows that (1) $R(g^2, k^2)$ is analytic in the cut k^2 plane and has a pole at k_0^2, and (2) $R(g^2, k^2) \to 0$ as $k^2 \to \infty$.

Therefore, a straightforward application of Cauchy's theorem yields

$$R(g^2, k^2) = \frac{C}{k^2-k_0^2} + \int_{a^2}^{\infty} du^2 \frac{P_1(g^2, u^2)}{(k^2+u^2)}, \tag{4}$$

† The main results of the present paper remain valid if subtractions are included into the axiomatic representation.

where C is minus the residue of Δ'_F at k_0^2 and P_1 is a provisionally arbitrary function, but such that the integral on the right-hand side of eq. (4) exists.

One now easily obtains from eqs. (1)-(4)

$$K(g^2, u^2) = \frac{\operatorname{Im}F(g^2, -u^2-i\varepsilon)}{\pi(m^2-u^2)[(1-\operatorname{Re}F(g^2,-u^2-i\varepsilon))^2+(\operatorname{Im}F(g^2,-u^2-i\varepsilon))^2]} + P_1(g^2, u^2). \quad (5)$$

Since K is real, so also must be P_1.

From (2), multiplying both sides by k^2 and taking the limit $k^2 \to \infty$, one finds with (4) that

$$C = 1 + \int_{a^2}^{\infty} \mathrm{d}u^2\{K(g^2, u^2) - P_1(g^2, u^2)\}. \quad (6)$$

Hence, C is real.

From the imaginary part of (4), for $k^2 = -u^2-i\varepsilon$, one finds

$$\operatorname{Im}R(g^2, -u^2-i\varepsilon) = P_1(g^2, u^2). \quad (7)$$

Consequently, P_1 must have zero asymptotic expansion in the coupling constant.

From (5) and (7) it follows that $\operatorname{Im}F(g^2, -u^2-i\varepsilon) \leqq 0$ for any given u^2 and sufficiently small $g^2 > 0$, since $K \geqq 0$ and P_1 vanishes faster than any power of g^2. This imposes an additional restriction on the function F.

The requirement that K must be a regular function of g^2, or equivalently, that the discontinuities of Δ'_F and $\bar{\Delta}'_F$ coincide across the cut, implies $P_1 \equiv 0$. This can be true only if $\operatorname{Im}F(g^2, -u^2-i\varepsilon) \leqq 0$ for $g^2 > 0$. The propagator $\bar{\Delta}'_F$ so obtained (with $P_1 \equiv 0$) coincides with the one proposed by Redmond [2]) and Bogolyubov *et al.* [3]).

3. Behaviour of the Ghost Pole

We now discuss the additional restrictions on the function F arising from the requirement of asymptotic equivalence.

Since generally the asymptotic vanishing of P_1 ensures the same behaviour for its Stieltjes transform in (4), it will follow that, if no further specifications about P_1 are made, the term $C(g^2)/(k^2-k_0^2(g^2))$ in (4) must also be asymptotically zero. This implies that for $g^2 \to +0$ either $C(g^2) \to 0$ faster than any power of g^2, or that $k_0^2(g^2) \to \infty$ faster than any inverse power of g^2, or both. We shall see below that under our initial assumptions we must have $k_0^2(g^2) \to \infty$ faster than any inverse power of g^2.

For the sake of simplicity we put $P_1 \equiv 0$ in what follows.

To obtain a spectral representation for F we observe that from

$$\frac{1}{(k^2+m^2)(1-F(g^2,k^2))} = \frac{1}{k^2+m^2} + \int_{a^2}^{\infty} du^2 \frac{K(g^2,u^2)}{k^2+u^2} - \frac{C}{k^2-k_0^2} \tag{8}$$

and eq. (6) we have

$$\frac{1}{1-F(g^2,k^2)} = \int_{a^2}^{\infty} du^2 \frac{K(g^2,u^2)(m^2-u^2)}{k^2+u^2} - \frac{k_0^2+m^2}{k^2-k_0^2}\left\{\int_{a^2}^{\infty} du^2 K(g^2,u^2)+1\right\}. \tag{9}$$

Therefore, from well known properties of the Stieltjes transform, it follows that for $g^2 > 0$

$$\lim_{|k^2|\to\infty} F(g^2,k^2)/|k^2| \neq 0 \text{ and finite} \tag{10a}$$

if $\int_{a^2}^{\infty} du^2 K(g^2,u^2)(u^2-m^2)$ is finite, or

$$\lim_{|k^2|\to\infty} F(g^2,k^2)/k^2 = 0 \tag{10b}$$

if $\int_{a^2}^{\infty} du^2 K(g^2,u^2)(u^2-m^2)$ is infinite.

In any case Cauchy's theorem can be applied to $F(g^2,k^2)/(k^2+m^2)^2$, the infinite circle giving zero contribution. Recalling that $F(g^2,-m^2)=0$, one finds

$$F(g^2,k^2) = A(g^2)(k^2+m^2) + \left\{\int_{a^2}^{\infty} du^2 P(g^2,u^2)/(k^2+u^2)\right\}(k^2+m^2)^2, \tag{11}$$

what also can be written as

$$F(g^2,k^2) = B(g^2)(k^2+m^2) + \left\{\int_{a^2}^{\infty} du^2 P'(g^2,u^2)/(k^2+u^2)\right\}(k^2+m^2). \tag{12}$$

Although calculated for $g^2 > 0$ such a representation would hold for any g^2 inside the analyticity domain of F as a function of g^2, by continuation in g^2 *under* the integral sign. For definiteness we shall restrict ourselves to such functions F for which this continuation coincides with the analytic continuation of F itself †; in this case F would increase not faster than k^2 for *all* g^2 in the analyticity domain. For what follows however, it would have been sufficient to consider functions F which do not increase faster than a certain power of k^2, for all g^2 in the analyticity domain.

For $g^2 > 0$, one has $P' \geqq 0$ in virtue of $\mathrm{Im} F(g^2, -u^2-i\varepsilon) \leqq 0$, and $\int_{a^2}^{\infty} du^2 P(g^2,u^2)$ is finite because of the rate of increase of F.

† This will be the case if for instance $F(g^2,k^2) = \lim_{n\to\infty} F'_n(g^2,k^2)$ for all g^2 in the analyticity domain, where $F'_n(g^2,k^2)$ has a cut extending from $-a^2$ to $-na^2$. As a simple counterexample of a function satisfying all our requirements and whose continuation cannot be defined by (12) we have

$$F(g^2,k^2) = g^2\left[\exp\left\{(1-g^2)\ln\frac{k^2+a^2}{a^2-m^2}\right\}-1\right]$$

for $\mathrm{Re}\, g^2 < 1$. Its continuation up to $\mathrm{Re}\, g^2 > -1$ can be defined by (11).

If case (10b) holds, one has $B \equiv 0$, i.e.

$$F(g^2, k^2) = \left\{ \int_{a^2}^{\infty} du^2 P'(g^2, u^2)/(k^2+u^2) \right\} (k^2+m^2), \tag{13}$$

which is the form expected from perturbation theory. In this case $\int_{a^2}^{\infty} du^2 P'(g^2, u^2)$ is infinite since $\Delta'_F \to 0$ faster than $1/|k^2|$ as $k^2 \to \infty$.

We show now that if $k_0^2(g^2)$ does not tend to infinity faster than any inverse power of g^2, then $C(g^2)$ cannot go to zero faster than any power of g^2 as $g^2 \to +0$, what would imply that $R(g^2, k^2)$ could not be asymptotically zero, contrary to our hypothesis.

By the definition of C we have

$$\begin{aligned} C &= 1/(k_0^2+m^2)\{\partial F(g^2, k_0^2)/\partial k_0^2\} \\ &= -(dk_0^2/dg^2)/(k_0^2+m^2)\{\partial F(g^2, k_0^2)/\partial g^2\}. \end{aligned} \tag{14}$$

Therefore, if C would tend to zero faster than any power without k_0^2 tending to infinity faster than any inverse power, then $\partial F/\partial g^2$ had to increase faster than any inverse power. This will be proved to be impossible.

In any case, we know already that $k_0^2 \to \infty$ as $g^2 \to +0$, in virtue of $F(0, k^2) = 0$.

We can write, using the fact that F is analytic in g^2,

$$\partial F(g^2, k_0^2)/\partial g^2 = \frac{1}{2\pi i} \oint F(z, k_0^2)/(z-g^2)^2 dz. \tag{15}$$

Thus, we get

$$|\partial F(g^2, k_0^2)/\partial g^2| \leqq \frac{1}{2\pi} \oint |dz F(z, k_0^2)/(z-g^2)|. \tag{16}$$

For g^2 in a sufficiently small vicinity of $g^2 = 0$, we may use as path of integration the circle with radius r with centre at the origin. Therefore, considering g^2 which lie in a vicinity of radius $\frac{1}{2}r$ of the origin, we find

$$|\partial F(g^2, k^2)/\partial g^2| \leqq (2/\pi r) \int_0^{2\pi} d\theta |F(z, k_0^2)|. \tag{17}$$

From (12) we know that

$$\lim_{|k^2| \to \infty} |F(z, k^2)|/|k^2|$$

is finite.

Hence, we have

$$\lim_{g^2 \to +0} \frac{|\partial F(g^2, k_0^2)/\partial g^2|}{k_0^2} \leqq \lim_{k_0^2 \to \infty} (2/\pi r) \int_0^{2\pi} d\theta |F(z, k_0^2)|/|k_0^2|) < \infty. \tag{18}$$

From this it follows that for the asymptotic equivalence to hold we must have

$$\lim_{g^2 \to +0} k_0^2(g^2) g^{2s} = \infty \quad \text{for any} \quad s > 0. \tag{19}$$

In case $F(g^2, k^2)$ results from a finite order perturbation approximation to the self-energy, i.e. $F(g^2, k^2) = \sum_{n=1}^{N} g^{2n} F_n(k^2)$, the following conclusion may be easily drawn from (19): $\lim_{k^2\to\infty} F_n(k^2)/k^{2s} = 0$ for all n and any $s > 0$, and consequently $B(g^2) \equiv 0$ in (12).

Of course the same result holds if $F(g^2, k^2) = \sum_{n=1}^{\infty} g^{2n} F_n(k^2)$, with $F_n(k^2) \geqq 0$ for all n larger than a certain n_0 and k^2 larger than a fixed value.

From the preceding considerations we see that the requirement of asymptotic equivalence imposes a rather stringent condition upon the behaviour of F, i.e. upon the location of the ghost pole as a function of the coupling constant.

4. Physical Consequences of the Asymptotic Equivalence

We finally discuss the implications of the asymptotic equivalence by means of two examples:

1) the iterated bubble approximation, where $F(g^2, k^2) = g^2F_1(k^2)$, and 2) taking for F the more particular from $F(g^2, k^2) = g^2F_1(k^2\Phi(g^2))$, which is chosen by analogy with the result of Gell-Mann and Low's [5]) analysis of the high-energy behaviour of propagators in quantum electrodynamics. The first case is obviously included in this one by taking $\Phi \equiv 1$.

From

$$g^2F_1(k^2\Phi(g^2)) = 1, \tag{20}$$

one obtains

$$(1/g^2)+\{\mathrm{d}(k_0{}^2\Phi(g^2))/\mathrm{d}g^2\Phi(g^2)\}[\partial\{g^2F_1(k_0{}^2\Phi(g^2))\}/\partial k_0{}^2] = 0, \tag{21}$$

and from (14) and (21) it follows that

$$C = -g^2\{\mathrm{d}(k_0{}^2\Phi(g^2))/\mathrm{d}g^2\}(k_0{}^2+m^2)\Phi(g^2). \tag{22}$$

Since for $g^2 \to +0$, $k_0{}^2 \to \infty$ faster than any inverse power, and $\Phi(g^2)$ is a regular function in the vicinity of $g^2 = 0$, one finds

$$\lim_{g^2\to+0} C(g^2) = \infty. \tag{23}$$

This is in no contradiction with the asymptotic requirement since $k_0{}^2$ goes to infinity sufficiently fast to render $R(g^2, k^2)$ asymptotically zero.

Let us take $C(g^2) \approx rA'/g^{2r}$, with $A' > 0$, $r > 0$, for $g^2 \approx 0$. Then, by integration of (22) we obtain, putting $(k_0{}^2+m^2)/k_0{}^2 \approx 1$ for $g^2 \approx 0$, the result

$$k_0{}^2(g^2)\Phi(g^2) \approx B' \exp{(A'/g^{2r})} \qquad \text{for } g^2 \approx 0. \tag{24}$$

From this it follows that

$$F_1(k^2\Phi(g^2)) \approx (A')^{-1/r}\,[\log{(k^2\Phi(g^2)/B')}]^{1/r} \text{ for } k^2 \to \infty. \tag{25}$$

One could of course take C more singular than any inverse power of g^2, corresponding to a lower increase of F for large k^2, e.g. $F_1(k^2\Phi(g^2)) \approx$ loglog $(k^2\Phi(g^2))$, etc.

In case $m = 0$, the function F_1 can easily be expressed as a function of C according to

$$F_1(k^2\Phi(g^2)) = G[\log(k^2\Phi(g^2)/b)], \tag{26}$$

where $G^{-1}[x] = \int_1^x dt C(1/t)/t$ and b is an integration constant. The function F_1 satisfies the relations

$$\lim_{y\to\infty} (\log F_1(y))/\log y = 0, \qquad \lim_{y\to\infty} F_1(y) = \infty, \tag{27}$$

implying that $\lim_{y\to\infty} F_1(y)/y^s = 0$ for any $s > 0$.

From (6), with $P_1 \equiv 0$, we see that $C = Z_3^{-1}$ where Z_3 (to be called simply Z) is the renormalization constant. Using now the relation $g^2 = Z^{-1}g_0^2$, valid in quantum electrodynamics, one can discuss the general behaviour of the unrenormalized coupling constant as a function of the renormalized one.

From (23) it is seen that the 'axiomatized' theory will be singular at zero coupling: $Z^{-1}(g^2 \to +0) = \infty$.

For $r < 1$, g_0^2 tends to zero with g^2, while for $r > 1$, g_0^2 goes to infinity as g^2 tends to zero and, as $Z^{-1} \geqq 1$, the latter curve $g_0^2(g^2)$ must necessarily have a minimum at some finite $g_M^2 > 0$.

For $r = 1$, g_0^2 goes to the constant A' as g^2 tends to zero. In the case of the iterated bubble approximation of quantum electrodynamics and meson theory as well as in the Lee Model, one finds that $r = 1$, and calculation shows that g_0^2 increases with increasing g^2, and $g_0^2 \to A'$ as $g^2 \to +0$ in an asymptotic way, that is, $d^n g_0^2/dg^{2n} \to 0$ for $g^2 \to +0$ and $n \geqq 1$. If with increasing g^2, g_0^2 would start to decrease, the curve $g_0^2(g^2)$ would have to pass through a minimum at some $g_M^2 < A'$, as can be seen from $Z^{-1} \geqq 1$.

The behaviour of $g_0^2(g^2)$ for large g^2 was already studied in (I).

5. Conclusions

(a) The physical implications of a field theory with singular structure at zero coupling were discussed in detail in (I). In particular, the relation between bare and dressed particles was shown to differ from the usually accepted one.

(b) The possibility of obtaining finite relations between g and g_0 supports the hope that the theory might lead to definite numerical values such as observed charge or mass-charge ratios if supplemented with some additional conditions.

(c) If one assumes that in quantum-electrodynamics the exact propagator has a ghost pole and coincides asymptotically with an axiomatic one, then one of the results of Gell-Mann and Low's analysis, namely the fact that g_0 can be a finite non-zero constant independent of g could be weakened to imply that g_0

tends asymptotically to a constant as g tends to zero, corresponding to the case $r = 1$. Since this is also the case for the iterated bubble approximation, one would conclude that this approximation suffices to describe the essential qualitative features of the exact propagator for small g, even in the high energy region.

(d) From (10) one can draw the following conclusion: if perturbation theory leads to results of the type (13) and not of the general type (12), the integral $\int_{a^2}^{\infty} du^2 (u^2 - m^2) K(g^2, u^2)$ must be infinite. Since this integral is connected with boson mass renormalization one concludes that the boson self-mass in the modified theory will diverge. This conclusion, however, is not necessarily valid for the fermion self-mass since in that case the lower degree of divergency in perturbation theory will reflect itself also in the modified ghost-subtracted, and therefore axiomatic theory.

The author wishes to thank Prof. W. Heisenberg for his interest and for the hospitality at his Institute. He expresses his gratitude to Prof. W. Güttinger for many stimulating discussions and suggestions. He is grateful to Prof. M. Schönberg for the hospitality at his Institute, where most of this work was done under a grant of the Brazilian National Research Council. The grant of a Fellowship through the International Atomic Energy Agency is gratefully acknowledged.

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